Life as a Mathematician

Arthur Jaffe

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Coming to this extraordinarily impressive tournament has given me a wonderful introduction of how it is possible to have a mathematics competition that begins to bridge the gap between an ordinary test of "athletic" skill in mathematics, to the "real-life" of a professional mathematician. First I congratulate every one of the contestants for her or his performance, a small part of which I observed today. I address these remarks to you students, who are at a crucial juncture in your lives.

Life is a walk, *but it is not random*. It's up to you to make the most of your future. Clearly all of you have the native intelligence and ability to excel, and personally, I hope that you will choose a career in mathematics or science. As you know, discovery of mathematics or scientific knowledge brings tremendous elation and satisfaction to you individually. But it is also important for our lives. And above all, mathematical discovery represents a highpoint of human culture, like writing great literature or music, or painting great art. So as you contemplate how you can contribute to bettering the world, I humbly offer you some precepts garnered from reflections on my own career.

Precept 1: Be Modest
Many students regard real-life as a competition. However we live in a society, and in mathematics and science there is also a culture and personal interaction. One appreciates your speaking well of interesting ideas, work, and advances that are not yours—especially if made by your competitors, or if your work (as it always does) stems from the work before yours. So leave it to others to promote your own work, and focus on its content.

Precept 2: Respect those who helped you.
You are certainly indebted to your parents and to your teachers more than you think. It is better to be aware of this early in life. For then you can demonstrate your appreciation, and it will be acknowledged.

Leading up to this competition, you were given many interesting problems to consider for this tournament. In fact, formulating the right mathematical questions to ask in the correct areas to investigate turns out to be the *most difficult aspect* of mathematics.

Don't hesitate to take advice from your *teachers*, as well as your contemporaries. Your teachers generally know much more than you about what are the most important questions, what is known about them. Crucially they can tell you *what is not known*.

Precept 3: Respect Science
Research in mathematics is inevitably frustrating. Real life mathematics or real science is very different from competitions. You had a taste of this in the tournament in Bremen.

For in most other competitions, you work on questions that have already been understood. And real life is very different. When you arrive at a good mathematical question, you never know if it is possible to solve it.

This has different facets. First is the question whether your abilities, or your knowledge, are sufficient to tackle the problem. In fact sometimes it is an advantage to be naïve, and not to know all the failures that came earlier. But you also do not even know whether the difficulty of the problem transcends what any person can achieve in a lifetime. For example, the mathematician Papakyriakopoulos in Princeton dedicated his entire life to the Poincaré conjecture, without solving it. But during his lifetime he became known as the person who could find the flaw in anyone's proposed solution, as he had generally tried those methods earlier and knew their pitfalls.

And worse, you do not even know whether a given problem *can* be solved within the framework of conventional mathematical logic, as Gödel showed that there are problems which are undecidable.

A good scientist or mathematician must develop intuition to select problems that are difficult enough to be interesting, yet easy enough that one can find a solution. This is a combination of talent, imagination, inspiration, and practice, but *most of all the result of very hard work*. And the solution of these problems inevitably involves insight as well as frustration from false starts. One requires persistence. But most of all, one requires *much, much hard work!* I know of no successful mathematician who was not a workaholic.

Precept 4: Keep an Open Mind; Remain a Student

Everyone has his or her favorite type of mathematics: number theory vs. geometry, algebra vs. analysis, topology vs. heat flow, operator theory vs probability theory. But you should keep in mind the fact that many major advances in recent years come from combining ideas from different fields. This includes my own work in showing the mathematical compatibility of special relativity and quantum theory with interaction. Not only does the subject need operator theory, probability theory, algebra, topology, and geometry, but it seems to touch many areas of frontier research in these and other directions.

Another example is Gregory Perelman's proof of the 100-year old Poincaré conjecture, that brings together topology and heat flows, together with striking ideas from physics of renormalization flows and entropy. So one needs to think broadly and to look for good ideas in all directions.

What was my own story? My undergraduate degree is in chemistry at Princeton, during which time I expected to study medicine. I made the switch to mathematics when I had the opportunity to spend two years in Cambridge. Only there did I discover the fascinating papers by Arthur Wightman. So I decided I wanted to return to Princeton to work with someone I had never heard of during my time as an undergraduate in his university—in spite of the fact most of my friends studied mathematics or physics!

My advice is to pursue what fascinates you and keep an open mind. *A good mathematician remains a student throughout her or his entire life!*

Precept 5: Prepare for the Unexpected
Here is an example. After finishing in Cambridge, and before starting graduate school, I travelled from England to a summer school in Hercegnovi, Yugoslavia. After the lengthy trip from to Paris, I made my way to board the Orient Express for Trieste. To my dismay, the train was overflowing with people, many sitting on suitcases in the hallway, and my heart fell. I was exhausted and really wanted a place to rest. Going through the train as rapidly as I could, I spied what surely must have been the last empty seat. That problem was successfully solved.

But unexpectedly, to my great pleasure and surprise, I discovered that the person sitting next to me was a Frenchman, going to be a teacher at the very same school! What a coincidence! And it was so lucky that we struck up a conversation. I met Maurice Jacob, who became a life-long friend, and who eventually became the first diplomatic representative of CERN.

Mathematics holds many unexpected surprises. I was lucky to study mathematical physics. For in mathematics if anything can go wrong in a proof, it seems to do so. But physics seems to work for mathematics. If anything can go right, it will!!

So keep your eyes open.

Precept 6: Think Big
Solving a particular problem may actually be easier if one solves a problem that is more general than the one that you are working on. This is a very simple example: if A_{ij} and B_{ij} are entries of hermitian matrices with positive eigenvalues, so is the matrix with entries $C_{ij} = A_{ij}B_{ij}$. The easiest way to see that is to consider the tensor product of matrices, a more general structure, acting on a tensor product space. In particular one considers D as the product of $A \otimes I$ and $I \otimes B$. The positivity of this D is obvious, and the desired result follows from the fact that D agrees with C on the diagonal subspace of the tensor product.

Fermat's theorem has been solved, and so has the Poincaré conjecture, as well as the Kepler problem on close packing of spheres. But we still do not know that special relativity is compatible with quantum theory in our four-dimensional world. We still do not know whether one can build a quantum computer. We still do not know if the Goldbach conjecture is correct. The spectrum of interesting and important mathematical problems is too big to describe in one lecture.

In mathematics it is good to have your eye on some really important and difficult goal, hoping that you can make progress on it over time. But difficult problems generally don't succumb immediately, and it is important to have a palette of questions that you study of varying difficulty.

Precept 7: At the Same Time, Work Small
On the other hand, most progress proceeds in small steps. When tackle a big problem, you should break it into little pieces. The progress on the individual parts may appear slow, but the whole often is bigger than the sum of its parts—in the sense that the consequence of many small and obvious steps can lead to a very non-obvious conclusion! So do not neglect clearly understanding every small step.

Precept 8: Appreciate History
While I was a student in Princeton, Dirac gave a lecture in Oppenheimer's seminar about the history of the relativistic wave equation for the electron. At the end of the talk Oppenheimer asked, "Professor Dirac, at what point did you realize that your relativistic equation for the electron was correct?" Dirac replied, "I was working at home and had to go to the library to check the spectrum of hydrogen. But as I was getting my bicycle out of the garage, I realized that the equation was *so beautiful*, that it had to be right!" Beauty counts for a great deal!!

Precept 9: Always ask, "What's Next?"
Don't sit on your laurels. When you are successful, keep going! You should try to keep learning, discovering, and doing. When I showed the compatibility of relativity and quantum theory in two and three dimensions, I did not stop.
I went on to do extensive work in four or five other areas of mathematical physics.

Precept 10: Enjoy Life
Enjoy life. But keep in mind: events come and go; politicians come and go; businesses come and go; institutions come and go; and even countries come and go. *But mathematics lasts forever*.